On the Capacity of Memoryless Adversary

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***Abstract*—In this paper, we study a model of communication under adversarial noise. In this model, the adversary makes online decisions on whether to corrupt a transmitted bit based on only the value of that bit. Like the usual binary symmetric channel of information theory or the fully adversarial channel of combinatorial coding theory, the adversary can, with high probability, introduce at most a given fraction of error.**

It is shown that, the capacity (maximum rate of reliable information transfer) of such memoryless adversary is strictly below that of the binary symmetric channel. We give new upper bound on the capacity of such channel – the tightness of this upper bound remains an open question. The main component of our proof is the careful examination of error-correcting properties of a code with skewed distance distribution.

I. Introduction

Consider the usual definitions of discrete channels in information theory. It is assumed that, transmissions of symbols from a discrete alphabet take place and a fraction of the transmissions may result in erroneous reception. The sender is allowed to "encode" information in to an array of symbols, called a codeword. The collection of all possible codewords is called a "code" (or "codebook"). Without much loss of generality, we can assume that all transmitted codewords are equally likely, in which case the log-size of a code signify the amount of information that can be transmitted with the code. In a completely adversarial channel, the adversary is allowed to see the transmitted set of symbols (codeword) completely and then decides which of the transmitted symbols are to be corrupted (it is allowed to corrupt a given fraction of all symbols).

Recently, in a series of papers [8], [10], [12], the study of online or causal adversarial channels is initiated, in particular, for binary-input channels. Let us start by giving an informal definition of a causal adversarial channel. In the causal adversarial model, an adversary is allowed to see the transmitted codeword only causally (i.e., at any instance it sees only the past transmitted symbols), and decides whether to corrupt the current transmitted symbol. An upper bound on the capacity (maximum rate of reliable information transfer) of such channel is presented in [8]. One of the most interesting observation is that, such channels are limited by the "Plotkin bound," of coding theory: whenever the fraction of error introduced by the adversary surpasses $\frac{1}{4}$, the capacity is zero (assuming binary input). On the other hand, by "random coding" method, a lower bound is established in [10]. This lower bound beats the famous Gilbert-Varshamov bound, the best available lower bound for a completely adversarial channel.

We below describe an adversarial channel model that is weaker (in terms of adversary limitations) than the above causal channel. In particular, the adversary is not even allowed to see the past transmitted symbols, but decides whether to corrupt a symbol based on only the current transmission. Our initial aim is to see whether the channel capacity is still dictated by the Plotkin bound.

A. A memoryless (truly online) adversary

In this work we consider the code to be *deterministic*, in a sense that is described below. Also, we assume that the input alphabet to be binary ($\{0,1\}$). A code $\mathcal{C}$ is simply a subset of $\mathbb{F}_2^n$. The size of the code denotes the number of messages encodable with this code; and therefore the amount of information encodable is $\log|\mathcal{C}|$. In here and subsequently, all logarithms are base-2, unless otherwise mentioned. The *rate* of the code is $\frac{\log|\mathcal{C}|}{n}$.

Given the code, the adversarial channel consists of n (possibly random) functions $f_{\mathcal{C}}^i : \mathbb{F}_2 \to \mathbb{F}_2, i = 1,\ldots,n$. Suppose a randomly and uniformly chosen codeword

$$\boldsymbol{x} \equiv (x_1, x_2, \ldots, x_n) \in \mathcal{C}$$

is transmitted. At the ith time instant, the adversary will produce $e_i = f_{\mathcal{C}}^i(x_i)$, taking only the current transmitted symbol x_i as argument (and of course, taking into account the code $\mathcal{C}$, which is known to the adversary). Here, e_i is the indicator of an error at the ith position, $i = 1,\ldots,n$. I.e., the channel produces $y_i = x_i + e_i$, at the ith time-instance, where the addition is of course over $\mathbb{F}_2$.

Definition 1: The adversary is called *weakly*-p-*limited*, $0 \leqslant p \leqslant 1$, if the expected (with respect to the randomness in $f_{\mathcal{C}}^i$s and $\boldsymbol{x}$) Hamming weight of the error-vector $\boldsymbol{e} = (e_1, e_2, \ldots, e_n) = (f_{\mathcal{C}}^1(x_i), \ldots, f_{\mathcal{C}}^n(x_n)) \equiv f_{\mathcal{C}}(\boldsymbol{x})$ is

$$\mathbb{E}\,\mathrm{wt}(\boldsymbol{e}) \leqslant pn. \tag{1}$$

A more restrictive adversary (*strongly*-p-*limited*) must have,

$$\Pr(\mathrm{wt}(\boldsymbol{e})/n < p + \epsilon) = 1 - o(1), \forall \epsilon > 0. \tag{2}$$

A code is associated with a (possibly randomized) decoder $\phi : \mathbb{F}_2^n \to \mathcal{C}$. For a given pair of transmitted codeword and error vector, $\boldsymbol{x} \in \mathcal{C}, \boldsymbol{e} \in \mathbb{F}_2^n$, the decoder makes an error if,

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$\phi(\mathbf{x}+\mathbf{e}) \neq \mathbf{x}$. Given $\mathcal{C}$ and p, define $\mathrm{Adv}_w(\mathcal{C},p)$ to be the collection of all weakly-p-limited adversary strategies. That is, $f_{\mathcal{C}} \equiv \{f^i_{\mathcal{C}} : \mathbb{F}_2 \to \mathbb{F}_2, i = 1,\ldots,n\} \in \mathrm{Adv}_w(\mathcal{C},p)$ if and only if, $\mathbb{E}\,\mathrm{wt}(f_{\mathcal{C}}(\mathbf{x})) \leqslant pn$. Similarly, we can name the collection of all strongly-p-limited adversary strategies as $\mathrm{Adv}_s(\mathcal{C},p)$.

Our results, as in the case of causal adversarial channels of [12], holds for the case of *average probability of error* [1].

The average probability of error is defined to be,

$$P^w_{\mathcal{C}}(p) = \max_{f_{\mathcal{C}} \in \mathrm{Adv}_w(\mathcal{C},p)} \frac{1}{|\mathcal{C}|} \sum_{\mathbf{x}\in\mathcal{C}} \Pr(\phi(\mathbf{x}+f_{\mathcal{C}}(\mathbf{x})) \neq \mathbf{x}),$$

and,

$$P^s_{\mathcal{C}}(p) = \max_{f_{\mathcal{C}} \in \mathrm{Adv}_s(\mathcal{C},p)} \frac{1}{|\mathcal{C}|} \sum_{\mathbf{x}\in\mathcal{C}} \Pr(\phi(\mathbf{x}+f_{\mathcal{C}}(\mathbf{x})) \neq \mathbf{x}).$$

The maximum possible size of "good" codes are:

$$M^w_\epsilon(n,p) \equiv \max_{\mathcal{C}\subseteq\mathbb{F}_2^n : P^w_{\mathcal{C}}(p)\leqslant\epsilon} |\mathcal{C}|, \tag{3}$$

and,

$$M^s_\epsilon(n,p) \equiv \max_{\mathcal{C}\subseteq\mathbb{F}_2^n : P^s_{\mathcal{C}}(p)\leqslant\epsilon} |\mathcal{C}|. \tag{4}$$

Now, define the *capacities* to be,

$$C_w(p) \equiv \inf_{\epsilon>0} \limsup_{n\to\infty} \frac{\log M^w_\epsilon(n,p)}{n}, \tag{5}$$

$$C_s(p) \equiv \inf_{\epsilon>0} \limsup_{n\to\infty} \frac{\log M^s_\epsilon(n,p)}{n}. \tag{6}$$

It is evident that,

$$C_w(p) \leqslant C_s(p) \leqslant 1 - h_B(p), \tag{7}$$

where $h_B(x) = -x\log x - (1-x)\log(1-x)$ is the *binary entropy function*.

This is true because, a strongly-p-limited adversary strategy is to flip each symbol with probability p, independently. That is, the adversary can always simulate a binary symmetric channel, whose capacity is $1 - h_B(p)$.

B. Practical limitations to the model and contributions

It is counterintuitive to assume that the adversary, being memoryless, cannot store the previously transmitted bits, or its own actions, however, has access to the entire code and can do computations on them. But it should be noted that, the entire computation of the adversary is done offline, and in each transmission, it just performs according to one of the two options. Also note that, the adversary knows the time-instance of the transmission. That is, he knows that the ith transmission, among the n possible, is taking place. In that sense the adversary is not completely memoryless. The main purpose of introducing this model is to see how weak the adversary can be and still have its capacity dictated by the Plotkin bound.

On the other hand, the concept of such memoryless adversary appears in principle before in literature. In particular, general classes of restricted adversarial channels were considered in the literature of *arbitrarily varying channels* [2], [5], [6] or *oblivious channels* [11]. From [9, Thm. C.1] (see also, [1]), it is evident that the capacity of weakly-p-limited adversary is 0 for $p > \frac{1}{4}$. It is also proved there that, if the adversary can keep a count of how many bits it has flipped (a log-space channel), then the same fact holds for strongly limited adversaries as well.

In Sec. II, we present the above fact regarding weakly-limited adversary in a way that is amenable to our definitions. We then attempt to extend this result to the case of strongly-limited adversary, which forms the main contribution of this paper. In Sec. III we introduce the important notions of distance distribution of a code that proves useful in this context. In Sec. IV, we show that the capacity of a strongly-p-limited adversary is strictly separated from the capacity of a BSC(p). In particular we give an upper bound on $C_s(p)$ that is strictly below $1 - h_B(p)$ for all $p > \frac{1}{4}$. Further discussions and concluding remarks are presented in Sec. V.

II. Weakly-limited adversary

In this small section, we establish the following fact.

Theorem 1: $C_w(p) = 0$ for $p \geqslant \frac{1}{4}$.

To prove the theorem, the below lemma, known as the Plotkin bound, is used crucially.

Lemma 2 (Plotkin Bound): Suppose, $\mathcal{C} \subseteq \mathbb{F}_2^n$ is the code and $|\mathcal{C}| = M$. Randomly and uniformly (with replacement) choose two codeword $\mathbf{x}_1, \mathbf{x}_2$ from $\mathcal{C}$. Then,

$$\mathbb{E}d_H(\mathbf{x}_1,\mathbf{x}_2) \leqslant \frac{n}{2}, \tag{8}$$

where $d_H(\cdot)$ is the Hamming distance.

Proof: Consider an $M \times n$ matrix with the codewords of $\mathcal{C}$ as its rows. Suppose, λ_i is the number of 1s in the ith column of the matrix, $i = 1,\ldots,n$. Then,

$$\sum_{\mathbf{c}_1,\mathbf{c}_2\in\mathcal{C}} d_H(\mathbf{c}_1,\mathbf{c}_2) = 2\sum_{i=1}^{n} \lambda_i(M-\lambda_i) \leqslant \frac{nM^2}{2}.$$

Hence, $\mathbb{E}d_H(\mathbf{x}_1,\mathbf{x}_2) \leqslant \frac{n}{2}$, where, $\mathbf{x}_1,\mathbf{x}_2$ are two randomly and uniformly chosen codewords. ∎

Proof of Theorem 1: We show that there exists an adversary strategy that achieves the claim of the lemma. In this vein, we use the same adversarial strategy that is used in [8], [9]. Suppose, $\mathcal{C} \subset \mathbb{F}_2^n$ is the code and $|\mathcal{C}| = M$. The adversary (channel) first choses a codeword $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_2^n$ randomly and uniformly from $\mathcal{C}$. Now if $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ is the transmitted codeword, then,

$$e_i \equiv f^i_{\mathcal{C}}(x_i) = \begin{cases} 0, & \text{when } x_i = c_i \\ 1, & \text{with probability } \frac{1}{2} \text{ when } x_i \neq c_i \\ 0, & \text{with probability } \frac{1}{2} \text{ when } x_i \neq c_i. \end{cases}$$

[1] It is relatively easy to see that the worst-case probability of error does not lead to anything different than the completely adversarial channel. For the same reason linear codes do not lead to any improvement for these channels over completely adversarial channel. We refer to [8] for further discussion. In general, the notion of average vs. worst-case error probability leading to different capacities for *arbitrarily varying channels* is well-known (for example, see [2] or [13]).

Note that, if $\mathbf{c}$ is randomly and uniformly chosen from $\mathcal{C}$, then

$$\mathbb{E}\,\mathrm{wt}(\boldsymbol{e}) = \sum_{i=1}^{n} \Pr(e_i = 1) = \frac{1}{2}\sum_{i=1}^{n} \Pr(x_i \neq c_i) = \frac{1}{2}\mathbb{E}d_H(\mathbf{x}, \mathbf{c}) \leqslant \frac{n}{4},$$

where, $\boldsymbol{e} = (e_1, \ldots, e_n)$. Hence, the adversary is weakly-$\frac{1}{4}$-limited.

On the other hand, $\Pr(\mathbf{x} = \mathbf{c}) = \frac{1}{M}$. Suppose, $\mathbf{y} = \mathbf{x} + \boldsymbol{e}$. At the decoder, let $\Pr(\mathbf{y} \mid \mathbf{c}')$, $\mathbf{c}' \in \mathcal{C}$, denote the probability that $\mathbf{c}'$ is transmitted and $\mathbf{y}$ is received. Clearly,

$$\Pr(\mathbf{y} \mid \mathbf{c}) = \Pr(\mathbf{y} \mid \mathbf{x}).$$

Hence, even the maximum likelihood decoder will have a probability of error $\geqslant 1/2 - \frac{1}{M}$. Therefore, $C_w(p) = 0$ for $p \geqslant 1/4$. ∎

III. Distance distribution

To extend Thm. 1 to the case of strongly-limited adversary, we need to show an adversary strategy, that, with high probability, keep the number of errors within pn. However, for the adversary strategy of Thm. 1 to do this, we need the result of Lemma 2 to be stronger, i.e., a high probability statement. Let us now introduce some notations that help us cast Lemma 2 as a high-probability result.

The *distance distribution* of a code is defined in the following way. Suppose, $\mathcal{C} \subseteq \mathbb{F}_2^n$ be a code. Let, for $i = 0, 1, 2, \ldots, n$,

$$A_i = \frac{1}{|\mathcal{C}|}|\{(\mathbf{c}_1, \mathbf{c}_2) \in \mathcal{C}^2 : d_H(\mathbf{c}_1, \mathbf{c}_2) = i\}|. \tag{9}$$

As can be seen, $A_0 = 1$.

The dual distance distribution of a code is defined to be, for $i = 0, 1, \ldots, n$,

$$A_i^{\perp} = \frac{1}{|\mathcal{C}|}\sum_{j=0}^{n} K_i(j) A_j, \tag{10}$$

where

$$K_i(j) = \sum_{k=0}^{i} (-1)^k \binom{j}{k}\binom{n-j}{i-k}$$

is the Krawtchouk polynomial. Note that, $A_0^{\perp} = 1$. It is known that $A_i^{\perp} \geqslant 0$ for all i. The *dual distance* $d^{\perp}$ of the code is defined to be the smallest $i > 0$ such that $A_i^{\perp}$ nonzero.

Lemma 3 (Pless power moments): For all $r < d^{\perp}$,

$$\frac{1}{|\mathcal{C}|}\sum_{i=0}^{n}(n/2 - i)^r A_i = \frac{1}{2^n}\sum_{i=0}^{n}(n/2 - i)^r \binom{n}{i}. \tag{11}$$

Proof: For a proof of the lemma, see [14, p. 132]. ∎

Lemma 4: Suppose, $\mathcal{C} \subseteq \mathbb{F}_2^n$ is the code with dual distance greater than 2, and $|\mathcal{C}| = M$. Randomly and uniformly (with replacement) choose two codeword $\mathbf{x}_1, \mathbf{x}_2$ from $\mathcal{C}$. Then,

$$\Pr\Big(d_H(\mathbf{x}_1, \mathbf{x}_2) < n(1/2 + \epsilon)\Big) > 1 - \frac{1}{4n\epsilon^2}. \tag{12}$$

Proof: From Lemma 3, for any $r < d^{\perp}$,

$$\Pr\Big(d_H(\mathbf{x}_1, \mathbf{x}_2) \geqslant n(1/2 + \epsilon)\Big) \leqslant \frac{\mathbb{E}(d_H(\mathbf{x}_1, \mathbf{x}_2) - n/2)^r}{n^r\epsilon^r} = \frac{\frac{1}{2^n}\sum_{i=0}^{n}(n/2 - i)^r\binom{n}{i}}{n^r\epsilon^r}.$$

In particular, substituting $r = 2$ we have,

$$\Pr\Big(d_H(\mathbf{x}_1, \mathbf{x}_2) \geqslant n(1/2 + \epsilon)\Big) \leqslant \frac{n/4}{n^2\epsilon^2} = \frac{1}{4n\epsilon^2}.$$

∎

The implication of the above result is following. For any code $\mathcal{C}$ with dual distance greater than 2, there exists a strongly-p-limited adversary strategy such that, probability of error is at least $\frac{1}{2} - \frac{1}{|\mathcal{C}|}$ for all $p \geqslant \frac{1}{4}$. The proof follows along the lines of Thm. 1. However, this does not mean that the capacity of strongly-p-limited adversary becomes 0 for $p > \frac{1}{4}$. There may exist a code with dual distance less than or equal to 2 that can reliably transfer information at a nonzero rate for $p > \frac{1}{4}$. On the other hand, if the dual distance is that small, then the code must have a skewed or asymmetric distance distribution. In the next section, we will (formally) see that this fact forces the capacity of the strongly limited adversary to be strictly below that of binary-symmetric channel[2].

IV. Strongly-limited adversary

The main result of the paper concerns the capacity of strongly limited adversary and is given in the following theorem.

Theorem 5:

$$C_s(p) \leqslant \begin{cases} 1 - h_B(p), & p \leqslant \frac{1}{4} \\ h_B(1 - 3p + 4p^2) - h_B(p), & \frac{1}{4} < p \leqslant \frac{1}{2}. \end{cases} \tag{13}$$

To show this, we need to show the existence of an apt adversarial strategy.

A. The adversary strategy

The adversary uses the following strategy.

- $p \leqslant \frac{1}{4}$. The adversary just randomly and independently flips every bit with probability p.
- $p > \frac{1}{4}$. For the used code $\mathcal{C}$, the adversary calculates $L_{\mathcal{C}}(p, n) = \sum_{w > 2pn} A_w$, where A_w is the distance distribution of the code. The following two cases may occur.
  1) $\frac{L_{\mathcal{C}}(p,n)}{|\mathcal{C}|} = o(1)$. This case can be tested [3] if for any absolute constant ϵ, $\frac{L_{\mathcal{C}}(p,n)}{|\mathcal{C}|} < \epsilon$ for sufficiently large n. In this case, the adversary first choses a codeword $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_2^n$ randomly and uniformly from $\mathcal{C}$. Now if $\mathbf{c} = (c_1, c_2, \ldots, c_n)$

[2] It is known that the distribution of symbols (and even higher order strings) in the codebook needs to be *close* to the mutual information maximizing input distribution, such as uniform in BSC, for the code to achieve capacity (see [16]). However, distance distribution is different than input distribution; and we also want to quantify the gap to capacity.

[3] Indeed, whenever we talk about a code, we mean a code-family, that is indexed by n, the length. In this case, the adversary knows this code family. There is a way to bypass the $o(\cdot)$ notation, that we omit here for clarity.

is the transmitted codeword, then, errors are introduced in the following way

$$e_i \equiv f^i_{\mathcal{C}}(x_i) = \begin{cases} 0, & \text{when } x_i = c_i \\ 1, & \text{with Prob. } \frac{1}{2} \text{ when } x_i \neq c_i \\ 0, & \text{with Prob. } \frac{1}{2} \text{ when } x_i \neq c_i. \end{cases}$$

Let, $\boldsymbol{e} = (e_1, e_2, \ldots, e_n)$. The received codeword is $\mathbf{c} + \boldsymbol{e}$.

2) $\frac{L_{\mathcal{C}}(p,n)}{|\mathcal{C}|} \geqslant c$ for some absolute constant c for all n. In this case, the adversary just randomly and independently flips every bit with probability p.

B. Proof of Thm. 5

The following lemma will be useful in proving the theorem.

Lemma 6 (Capacity of constrained input): Let $R^*(p, \omega)$ denote the supremum of all achievable rates for a code (of length n) as $n \to \infty$ such that:

1) Each codeword has Hamming weight at most ωn, $\omega \leqslant \frac{1}{2}$.
2) The average probability of error of using this code over BSC(p) goes to 0 as $n \to \infty$.

Then

$$R^*(p, \omega) = h_B(\omega * p) - h_B(p),$$

where $\omega * p = (1 - \omega)p + \omega(1 - p)$.

Sketch of proof: To prove this lemma, we calculate the mutual information between the input and output of the BSC(p), when the inputs are i.i.d. Bernoulli(ω) random variables. It is not difficult to show that, such random code must contain almost as large a subset with weight of all codewords less than or equal to ωn. The converse follows from an application of Fano's inequality and noting that, asymptotically, $\log \binom{n}{\lambda n} \approx n h_B(\lambda)$. ∎

Proof of Thm. 5: If $p \leqslant \frac{1}{4}$ then the adversary just simulates the binary symmetric channel. Below we consider the situation when $p > \frac{1}{4}$.

In what follows, we treat the two different scenarios for the adversary, based on the adversary strategy sketched above. Let $\mathcal{C}$ is the code that is used for transmission and $\{A_w\}$ is the distance distribution of the code, as usual.

Case 1: Let, $\mathbf{x}$ is the codeword adversary has initially chosen. Note that, if $\mathbf{c}$ is randomly and uniformly chosen from $\mathcal{C}$, then, the random variable $W = d_H(\mathbf{c}, \mathbf{x})$ is distributed according to $\{A_w/|\mathcal{C}|, w = 0, \ldots n\}$.

We have,

$$\Pr\Big(W > 2pn\Big) = o(1).$$

Using Chernoff bound,

$$\Pr\Big(\text{wt}(\boldsymbol{e}) \geqslant n(p + \epsilon) \Big| d_H(\mathbf{c}, \mathbf{x}) \leqslant 2pn \Big) \leqslant e^{-2n\epsilon^2}.$$

Hence, for any $\epsilon > 0$,

$$\Pr\Big(\text{wt}(\boldsymbol{e}) < n(p + \epsilon) \Big) > 1 - o(1),$$

which implies that the adversary is strongly-p-limited.

Now, just following the arguments of Thm. 1 we conclude that the code $\mathcal{C}$ will result in a probability of error at least $\frac{1}{2} - \frac{1}{M}$ with this adversary. Therefore, If $C_s(p) > 0$, then the next case must be satisfied for a code.

Case 2: In this case, there exists absolute constant $0 < c < 1$ such that,

$$\sum_{w > 2pn} A_w \geqslant c|\mathcal{C}|. \tag{14}$$

For any codeword $\mathbf{x} \in \mathcal{C}$, let $A^{\mathbf{x}}_w, w = 0, \ldots, n$ be the *local weight distribution*, i.e., the number of codewords that are at distance w from $\mathbf{x}$. Now as,

$$\sum_{w > 2pn} A_w = \frac{1}{|\mathcal{C}|} \sum_{\mathbf{x} \in \mathcal{C}} \Big(\sum_{w > 2pn} A^{\mathbf{x}}_w \Big),$$

it is clear that there must exist a codeword $\mathbf{x}$ such that

$$\sum_{w > 2pn} A^{\mathbf{x}}_w \geqslant c|\mathcal{C}|.$$

This ensures that, there are at least $c|\mathcal{C}|$ codewords that belong within a Hamming ball of radius $n - 2pn = n(1 - 2p)$. In particular, consider the ball of radius $n - 2pn$ centered at $\bar{\mathbf{x}}$, where $\bar{\mathbf{x}}$ is the complement of $\mathbf{c}$ (all zeros are changed to ones, and vice versa). All the codewords of $\mathcal{C}$ that are distance more than $2pn$ away from $\mathbf{x}$ must belong to this ball; let us call the set of such codewords $\mathcal{B} \subset \mathcal{C}$. Clearly $|\mathcal{B}| \geqslant c|\mathcal{C}|$.

Consider the average probability of error, when $\mathcal{B}$ is used to transmit a message over a BSC(p). Because, the Hamming space is translation invariant, the probability of error of such code is equal to the probability of error of a code $\hat{\mathcal{B}}$ that have the Hamming weight of each codeword bounded by $n(1-2p)$. But from Lemma 6, the maximum possible rate for which the probability of error of using $\mathcal{B}$ in BSC(p) goes to 0 is $R^*(p, 1 - 2p)$.

However, if we randomly pick up a codeword from $\mathcal{C}$, with probability at least $c > 0$, the codeword belong to $\mathcal{B}$. Hence $\frac{1}{n} \log |\mathcal{B}|$ must be less than $R^*(p, 1 - 2p)$, otherwise the average probability of error for $\mathcal{C}$ will be bounded away from 0. Hence, the rate of $\mathcal{C}$ is at most

$$R^*(p, 1 - 2p) = h_B(1 - 3p + 4p^2) - h_B(p).$$

∎

The capacity of strongly-limited adversary is strictly bounded away from the capacity of BSC. Indeed, $h_B(1-3p+4p^2) < 1$ for all $\frac{1}{4} < p < \frac{1}{2}$. This is shown in Figure 1.

C. Erasure Channel

The entire analysis of the above section can be extended for the case of a memoryless adversarial erasure channel, where instead of corrupting a symbol, the adversary introduces an erasure. Recently, an extension (that results in rather nontrivial

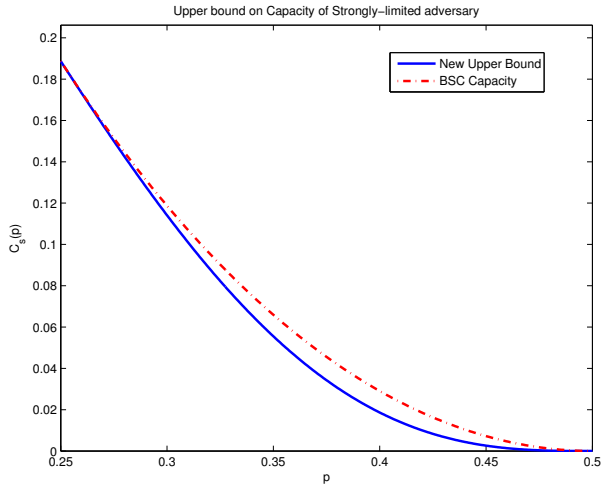

Fig. 1. The upper bound of Thm. 5 on the strongly-limited adversary.

observations) of the results of [8], [10] for the case of erasures have been performed in [3].

We refrain from formally defining a binary-input memoryless adversarial erasure channel; however, that can be done easily along the lines of the introductory discussions of this paper. For the case of weakly-p-limited adversary the capacity is zero for all $p \geqslant \frac{1}{2}$. On the other hand, we note that, for strongly-p-limited adversarial erasure channel the capacity is upper bounded by

$$(1-p)h_B(p),$$

for all $p \geqslant \frac{1}{2}$. The analysis is similar to that of this section, and uses the capacity of a constrained input erasure channel as a component of the proof (for example, see Eq. 7.15 of [4]).

V. A code with skewed distance distribution

In conclusion we outline a possible route through which an improvement on the upper bound on $C_s(p)$ might be possible.

From the proof of Thm. 5 it is evident that a code $\mathcal{C}$ that has nonzero rate can achieve a zero probability of error for the strongly-p-limited adversary only if the distance distribution $\{A_w, w = 0, \ldots, n\}$ satisfies, for some absolute constant $c > 0$,

$$\sum_{w > 2pn} A_w \geqslant c|\mathcal{C}|. \tag{15}$$

From, Delsarte's theory of linear-programming bounds [7], it is possible to upper bound the maximum possible size of such code $\mathcal{C}$. Indeed, this is given in the following theorem .

Theorem 7: Suppose, a code $\mathcal{C}$ is such that its distance distribution $\{A_w, w = 0, \ldots, n\}$ satisfies (15) for some $c > 0$. Assume there exist a polynomial $f(x)$ of degree at most n with,

$$f(x) = \sum_{k=0}^{n} f_k K_k(x), \tag{16}$$

and some $\beta > 0$, that satisfy,

1) $f_0 = 1$, $f_k \geqslant 0$ for $k = 1 \ldots, n$;
2) $f(j) \leqslant c\beta$ for $j = 1, \ldots, 2pn$ and $f(j) \leqslant -(1-c)\beta$ for $j = 2pn+1, \ldots, n$.

Then

$$|\mathcal{C}| \leqslant f(0) - c\beta.$$

Proof: We note that, $A_i^{\perp} \geqslant 0$ for all $i = 0, \ldots, n$, a set of linear constraints on the distance distribution whose sum we want to maximize. Moreover we have the extra linear constraint of (15). We omit the proof here, but if follows from standard arguments of linear programming bounds for codes. ■

If one could find a polynomial that satisfies the above conditions then that gives bounds on the capacity of strongly-p-limited adversary. Our current approach involves tweaking the existing polynomials that bound error-correcting codes (i.e., the MRRW polynomials [15]) to construct a polynomial that satisfy the criteria of Thm. 7.

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